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INTERSECTION PROPERTIES OF BOXES PART II: EXTREMAL FAMILIES

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ABSTRACT
In a previous paper (this Journal, Vol. 62, pp. 283-301) we proved an Upper-Bound Theorem for finite families of boxes in R^d with edges parallel to the coordinate axes. This theorem concerns the maximum possible numbers of intersecting subfamilies of a family having a given number of members and a given clique number. Here we give an intrinsic characterization of *extremal* families of boxes, i.e., families for which all the respective maximum numbers are achieved. We also deal with the problem of enumerating the possible intersection types of extremal families.

1. Introduction

In this paper we continue our investigation of intersection properties of boxes begun in [4]. Hereafter, that paper will be referred to as Part I and references to it will be preceded by the Roman numeral I.

In order to make the present article as self-contained as possible, we begin with a brief review of the relevant definitions and the principal result obtained in Part I.

A *box* in R^d is a cartesian product of d intervals, one on each of the d coordinate axes of R^d. By an *interval* we mean a non-empty closed convex set. In other words, a box is a closed convex parallelohedron, possibly unbounded, with edges (if any) parallel to the coordinate axes. Examples of unbounded boxes are the space R^d itself and any hyperplane parallel to one of the coordinate hyperplanes

$$H_i = \left\{ (x_1, \ldots, x_d) \in R^d \,\middle|\, x_i = 0 \right\}, \qquad i = 1, \ldots, d.$$

We shall be concerned with finite families of boxes in R^d. For such a family $\mathfrak{P}$, let $G(\mathfrak{P})$ denote the *intersection graph* of $\mathfrak{P}$, that is, the graph whose vertices are

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in one-to-one correspondence with the members of $\mathfrak{P}$ and in which two vertices are joined by an edge if, and only if, the corresponding members intersect. When $d = 1$, the resulting graphs are the familiar *interval graphs*. We shall say that two families of boxes $\mathfrak{P}$ and $\mathfrak{Q}$ are of the same *intersection type* provided the graphs $G(\mathfrak{P})$ and $G(\mathfrak{Q})$ are isomorphic. Since most of the intersection properties of boxes studied in this paper (and in Part I, for that matter) depend on $G(\mathfrak{P})$ rather than on $\mathfrak{P}$ itself, we shall regard families being of the same intersection type as not essentially distinct. Incidentally, it is easily seen that for each finite family of boxes in R^d there is a family of *compact* boxes with the same intersection graph. So it would be enough, for our purpose, to assume that boxes are parallelotopes with edges parallel to the coordinate axes. However, to allow for unbounded sets is occasionally convenient; compare, for example, the definition of a Kalai family below.

For a family of boxes $\mathfrak{P}$ and a non-negative integer k, we let $f_k(\mathfrak{P})$ denote the number of subfamilies of $\mathfrak{P}$ of size $k+1$ having non-empty intersection. Equivalently, $f_k(\mathfrak{P})$ is the number of $(k+1)$-cliques in $G(\mathfrak{P})$. (All graph terminology is explained at the end of this Introduction.) The sequence $f(\mathfrak{P}) = (f_0(\mathfrak{P}), f_1(\mathfrak{P}), f_2(\mathfrak{P}), \ldots)$ is called the *f-vector* of $\mathfrak{P}$. Our main objective in Part I was to establish tight upper bounds for the numbers $f_k(\mathfrak{P})$, given that $\mathfrak{P}$ has n members and no $r+1$ members of $\mathfrak{P}$ have a common point (i.e., $f_0(\mathfrak{P}) = n$ and $f_r(\mathfrak{P}) = 0$). Here n and r are fixed positive integers. The present paper differs from Part I by assuming that $n \geqq r \geqq d$.

We need to recall the definition of the family $\mathfrak{C}(n,d,r)$ which was introduced by Kalai [6] and will be called a *Kalai family*. Partition the number $n - r + d$ into d almost equal integer parts $n_1, \ldots, n_d$, say. Then $\mathfrak{C}(n,d,r)$ consists of n_i distinct translates of H_i for $i = 1, \ldots, d$, plus $r - d$ copies of R^d. Note that $f_0(\mathfrak{C}(n,d,r)) = n$ and $f_r(\mathfrak{C}(n,d,r)) = 0$. The main result of Part I, Theorem I.3.2, can now be stated as follows.

THEOREM 1.1. *Let $\mathfrak{P}$ be a family of n boxes in R^d, and suppose $f_r(\mathfrak{P}) = 0$. Then, for $k = 1, \ldots, r-1$, $f_k(\mathfrak{P}) \leqq f_k(\mathfrak{C}(n,d,r))$. Moreover, if equality is attained for some $k \in \{d, \ldots, r-1\}$, then equality is attained for each $k \in \{1, \ldots, r-1\}$.*

That the intersection graph of $\mathfrak{P}$ cannot have more than $f_1(n,d,r)$ edges had been conjectured earlier by Kalai [6].

Set $f(n,d,r) = f(\mathfrak{C}(n,d,r))$ and for each $k \geqq 0$, $f_k(n,d,r) = f_k(\mathfrak{C}(n,d,r))$. An explicit formula for these numbers is readily obtained. Write $n - r + d = pd + q$, where p and q are integers such that $0 \leqq q < d$. Then q of the parts used above in defining $\mathfrak{C}(n,d,r)$ equal $p+1$ and the remaining $d-q$ parts equal p. This yields (see (I.3.3) and (I.3.4))

$$f_k(n,d,r) = \sum_{j=0}^{d} \binom{r-d}{r-j+1} \sum_{i=0}^{j} \binom{q}{i} \binom{d-q}{j-i} (p+1)^i p^{j-i}. \tag{1.1}$$

Theorem 1.1 is an instance of what is termed an Upper-Bound Theorem (UBT) in combinatorial geometry. Other examples are the well-known UBT for the f-vectors of convex polytopes in R^d due to McMullen [7], or the more recent UBT for the f-vectors of families of (arbitrary) convex sets in R^d proved independently by Kalai [6] and the author [3]. For a short discussion of this topic we refer the reader to Part I.

At the same time, Theorem 1.1 can be viewed as a Turán-type result for the intersection graphs of families of boxes in R^d. In fact, Turán's "extremal graph theorem" (Turán [11],[12]), or rather its generalization by Zykov [15], is essentially the graph-theoretic analogue of Theorem 1.1 in the case $r \leqq d$. This case will not be considered here; it is described in Section I.3. By contrast, the geometry of boxes plays a dominant role when $r \geqq d$. It will be seen that $\mathfrak{C}(n,d,r)$ is just one of various possible families of boxes for which the upper bounds of Theorem 1.1 are attained for each k. This motivates the following

Definition 1.2. A family of n boxes in R^d is called *extremal* if, for a suitable $r \in \{d,\ldots,n\}$, its f-vector equals $f(n,d,r)$.

The goal of this paper is to give an intrinsic characterization of extremal families of boxes in R^d (up to intersection type). Our main result is Theorem 2.1, which will be formulated and commented on in Section 2. Its proof has to be postponed till Section 4. In Section 3 we shall describe the remarkable geometric properties exhibited by extremal families. Some of these properties will be used in the inductive proof of Theorem 2.1. The final Section 5 deals with enumeration problems. A graph-theoretical lemma of Zykov [15] will enable us to compute (at least in principle) the number of intersection types of extremal families having a given f-vector.

We wish to point out that the problem of characterizing the class of "extremal" polytopes in McMullen's UBT is still open, and so is the problem of characterizing the class of "extremal" families of convex sets in the UBT of Kalai and the author.

It remains to lay down some notation and definitions which will be used throughout this paper. All graphs considered are finite and without loops or multiple edges. The complete graph on n vertices is denoted by K_n, and $K_{n_1,\dots,n_r}$ stands for the complete r-partite graph whose vertex classes have $n_1,\dots,n_r$ vertices, respectively. In case the n_i are as nearly equal as possible and $n = n_1 + \cdots + n_r$, the latter graph is the so-called *Turán graph* $T_r(n)$.

A set of mutually adjacent vertices of a graph G is called a *clique* in G, or a *k-clique* if it has k vertices. The largest number k for which G contains a k-clique is called the *clique number* of G. A vertex v of G is said to be *dominant* in G provided v is adjacent to any other vertex of G and the graph has at least two vertices. Adjacent vertices are also called *neighbors*. The complement of G is denoted by $\bar{G}$. Thus $\bar{K}_n$ is the totally disconnected graph on n vertices. We write $G \cong H$ to indicate that the graphs G and H are isomorphic. The *join* of G and H, denoted $G + H$, is defined only when G and H have no vertex in common. It is obtained from the union of G and H by adding all the possible edges between a vertex of G and a vertex of H. Clearly, the join operation on graphs is commutative and associative.

Finally, let $\mathfrak{P}$ be a family of sets. Extending definitions above, we shall say that two members A and B of $\mathfrak{P}$ are *neighbors* in $\mathfrak{P}$ provided they intersect, and we call *A dominant* in $\mathfrak{P}$ if A is a neighbor of any other member of $\mathfrak{P}$ and the family has at least two members.

2. The main theorem

In this section we present our principal result, i.e., Theorem 2.1 below. It describes the manner in which extremal families of boxes in R^d are built up from their one-dimensional counterparts. We illustrate the procedure with an example in the plane and remark upon some salient points in the general case. As mentioned earlier, the geometrical properties of extremal families will be discussed in Section 3, and the proof of Theorem 2.1 will be given in Section 4.

We must first consider the particular case $d = 1$. In this case, a box is simply an interval on the real line. The intersection graphs of extremal families of intervals were studied in the author's recent paper [5] and will be called *extremal interval graphs*. These graphs are the building blocks used in constructing extremal families in higher dimensions. In a sense to be made precise later, each extremal family of boxes in R^d is determined by a suitable collection of d extremal interval graphs, and the converse is also true.

The following brief discussion of extremal interval graphs is based on [5]. According to the definition given in Section 1, an interval graph on n vertices is extremal provided it has

$$f_1(n,1,r) = \binom{r}{2} + (r-1)(n-r)$$

edges, where r is its clique number. We have shown in [5] that these graphs form a special class of $(r-1)$-trees when $r > 1$ (see Beineke and Pippert [1]). They are generated from the complete graph K_r through a recursive procedure and can be characterized as being *interior* $(r-1)$*-caterpillars* in the sense of Proskurowski [8]. The ordinary caterpillars are obtained if $r = 2$, while K_n and $\bar{K}_n$ are the only extremal interval graphs for $r = n$ and $r = 1$, respectively.

We do not explain the meaning of the term "interior $(r-1)$-caterpillar". Rather, we list some properties of extremal families of intervals that stem directly from the inductive construction of such graphs. These properties will be used later on.

We begin with a definition. Suppose $\mathfrak{P}$ is a family of intervals on the line, $A \in \mathfrak{P}$, $A \neq R^1$. We say that A is *exposed* in $\mathfrak{P}$ if no member of $\mathfrak{P}$ has a smaller right endpoint, or no member has a larger left endpoint, or finally A is a single point. (For a generalization, see Section 3.) Now assume that $\mathfrak{P}$ is extremal, with $f(\mathfrak{P}) = f(n,1,r)$, say. If $n = r$, then by Helly's Theorem for the line there is a point common to all members of $\mathfrak{P}$. If, however, $n \geqq r+1$, then the following is true.

(2.1) *$\mathfrak{P}$ contains exactly two exposed members A and A', say. Moreover, $A \cap A' = \emptyset$. Each point (resp., endpoint) of the open interval separating A and A' lies in at least $r-1$ (resp., exactly r) members of $\mathfrak{P}$.*

This implies

(2.2) *If A is exposed in $\mathfrak{P}$, then $f(\mathfrak{P}\backslash A) = f(n-1,1,r)$ and hence $\mathfrak{P}\backslash A$ is extremal.*

In the opposite direction we have

(2.3) *Let A be exposed in $\mathfrak{P}$, and let B be an interval, $B \notin \mathfrak{P}$. Suppose B meets all the neighbors of A in $\mathfrak{P}$ but not A itself, or B meets A and all the neighbors of A in $\mathfrak{P}$ except one, and no other member of $\mathfrak{P}$. Then $f(\mathfrak{P} \cup B) = f(n+1,1,r)$ and hence $\mathfrak{P} \cup B$ is extremal.*

Finally, we note

(2.4) *$\mathfrak{P}$ contains a dominant member if $r > 1$ and $r > n/2$.*

Figure 1 illustrates different ways of passing from $\mathfrak{P}$ to $\mathfrak{P} \cup B$ in terms of the corresponding interval graph. (Here $n = 6$ and $r = 3$.) In general, B need not be exposed in $\mathfrak{P} \cup B$.

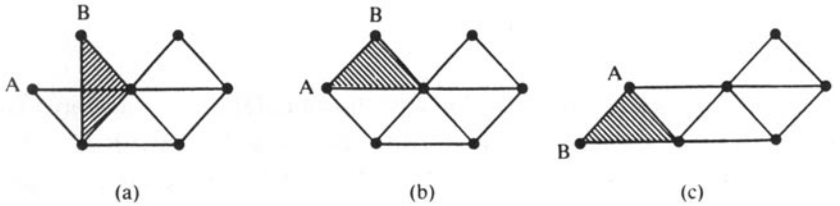

Fig. 1.

We are now ready to state our main result. As before, we assume that $n \geqq r \geqq d$ and define p and q by writing $n - r + d = pd + q$, $0 \leqq q < d$. For $i = 1, \ldots, d$, let τ_i denote the orthogonal projection of R^d onto the ith coordinate axis. Thus τ_i maps the point $(x_1, \ldots, x_d)$ to the point $(0, \ldots, 0, x_i, 0, \ldots, 0)$. If $\mathfrak{P}_i$ is a family of intervals on the x_i axis, then $\tau_i^{-1}(\mathfrak{P}_i) = \{\tau_i^{-1}(P) \mid P \in \mathfrak{P}_i\}$ is a family of (unbounded) boxes in R^d.

Then we have

THEOREM 2.1. (i) *For $i = 1, \ldots, d$, let $\mathfrak{P}_i$ be an extremal family of intervals on the x_i axis such that*

$$f(\mathfrak{P}_i) = \begin{cases} f(p + r_i, 1, r_i), & \text{if } i \in J, \\ f(p + r_i - 1, 1, r_i), & \text{if } i \notin J, \end{cases}$$

where the r_i are positive integers such that $r_1 + \cdots + r_d \leqq r$ and J is a set of q indices, $J \subset \{1, \ldots, d\}$. Suppose none of the $\mathfrak{P}_i$ contains a dominant member. Define the family $\mathfrak{P}_1 \times \cdots \times \mathfrak{P}_d$ in R^d to consist of all the boxes in

$$\bigcup_{i=1}^{d} \tau_i^{-1}(\mathfrak{P}_i)$$

as well as $r - (r_1 + \cdots + r_d)$ copies of R^d. Then $f(\mathfrak{P}_1 \times \cdots \times \mathfrak{P}_d) = f(n, d, r)$ and hence $\mathfrak{P}_1 \times \cdots \times \mathfrak{P}_d$ is extremal in R^d.

(ii) *Conversely, let $\mathfrak{P}$ be a family of boxes in R^d such that $f(\mathfrak{P}) = f(n, d, r)$. Then $\mathfrak{P}$ is of the type described above, that is, there exist families of intervals $\mathfrak{P}_1, \ldots, \mathfrak{P}_d$ satisfying the requirements made in* (i) *so that $G(\mathfrak{P}) \cong G(\mathfrak{P}_1 \times \cdots \times \mathfrak{P}_d)$. Moreover, $\mathfrak{P}_1, \ldots, \mathfrak{P}_d$ can be chosen in such a way that for each i and each interval $P_i \in \mathfrak{P}_i$, the box $P \in \mathfrak{P}$ corresponding to P_i under the isomorphism satisfies $\tau_i(P) = P_i$.*

Some comments regarding these statements are in order.

The families $\mathfrak{P}_1 \times \cdots \times \mathfrak{P}_d$ defined in part (i) of Theorem 2.1 are the prototype extremal families, in the sense that for each extremal family of boxes in R^d there exists a family of the form $\mathfrak{P}_1 \times \cdots \times \mathfrak{P}_d$ having the same intersection graph. Note that the members of $\mathfrak{P}_1 \times \cdots \times \mathfrak{P}_d$ are unbounded. However, it is straightforward to derive a family of *compact* boxes without changing the intersection type. Simply intersect each member with a fixed compact box which is large enough to meet all non-empty intersections of members of the family. (Compare also (3.5) below.) More generally, the only freedom we have in realizing the intersection type of $\mathfrak{P}_1 \times \cdots \times \mathfrak{P}_d$—apart from permuting the axes—is to intersect each member of $\mathfrak{P}_1 \times \cdots \times \mathfrak{P}_d$ with a suitable box which will vary with the individual member and may be bounded or unbounded. In essence, this is the meaning of part (ii) of Theorem 2.1.

The intersection type of $\mathfrak{P}_1 \times \cdots \times \mathfrak{P}_d$ is completely determined by the intersection types of $\mathfrak{P}_1, \ldots, \mathfrak{P}_d$ (justifying the notation in the first place). In fact, making use of the join operation on graphs, we find that

$$G(\mathfrak{P}_1 \times \cdots \times \mathfrak{P}_d) \cong G(\mathfrak{P}_1) + \cdots + G(\mathfrak{P}_d) + K_v,$$

where $v = r - (r_1 + \cdots + r_d)$ and the vertices of K_v represent the copies of R^d used in the construction. We shall see in Section 5 that the converse is also true. That is, if two extremal families $\mathfrak{P}_1 \times \cdots \times \mathfrak{P}_d$ and $\mathfrak{Q}_1 \times \cdots \times \mathfrak{Q}_d$ have isomorphic intersection graphs, then after renumbering (if necessary),

$$G(\mathfrak{P}_i) \cong G(\mathfrak{Q}_i), \qquad i = 1, \ldots, d.$$

This fact will enable us to enumerate the possible intersection types of extremal families.

We remark that the intersection graph of an extremal family of boxes is *perfect*, that is, each induced subgraph has the property that its clique number equals its chromatic number. Since interval graphs are perfect (as noted by Gallai), it is enough to observe that the join of perfect graphs is perfect. Actually, any maximal clique in $G(\mathfrak{P}_i)$ has size r_i, and so any maximal clique in $G(\mathfrak{P}_1 \times \cdots \times \mathfrak{P}_d)$ has size r. The clique numbers r_i satisfy $1 \leqq r_i \leqq r - d + 1$, $i = 1, \ldots, d$, with $r_i = r - d + 1$ for *some* i forcing $r_j = 1$ for $j \neq i$. If $r_i = 1$ for each i, then $\mathfrak{P}_i$ consists of $p + 1$ or p pairwise disjoint intervals, according as $i \in J$ or $i \notin J$; hence $\mathfrak{P}_1 \times \cdots \times \mathfrak{P}_d$ is of the same type as the Kalai family $\mathfrak{C}(n, d, r)$ defined in Section 1.

It follows, incidentally, that $\mathfrak{C}(n, d, r)$ is the only type of extremal family when $n \leqq r + d$. In this case, we have either $p = 1$ and $q = n - r$, or $p = 2$ and $q = 0$.

Since $G(\mathfrak{P}_i)$ does not have dominant vertices, assertion (2.4) implies that $r_i = 1$ for each i. Thus $G(\mathfrak{P}_1 \times \cdots \times \mathfrak{P}_d)$ is the Turán graph

$$T_r(n) = K_{2,\ldots,2,1,\ldots,1} \tag{2.5}$$

with $n - r$ 2's and $2r - n$ 1's. Note that $v = r - d$, whereas the number of dominant vertices in (2.5) is $2r - n$. However, as soon as $n \geqq r + d$ and hence $p \geqq 2$, each $G(\mathfrak{P}_i)$ has at least two vertices and so the dominant members of $\mathfrak{P}_1 \times \cdots \times \mathfrak{P}_d$ are precisely the $r - (r_1 + \cdots + r_d)$ copies of R^d belonging to the family.

We point out that (2.5) is the graph of the cross-polytope in R^d when $n = 2d$ and $r = d$. According to Roberts [9] (see also Wegner [13], Witsenhausen [14], and Trotter [10]), $T_d(2d)$ is the only graph on $2d$ or fewer vertices which is not the intersection graph of boxes in R^{d-1}.

Let us conclude this section by presenting one simple example in the plane.

Take $n = 9$, $d = 2$, and $r = 4$. What are the families of boxes in R^2 having f-vector $f(9,2,4) = (9,27,31,12,0,0,\ldots)$? Since $p = 3$ and $q = 1$, we may restrict ourselves to families of the form $\mathfrak{P}_1 \times \mathfrak{P}_2$, where $f(\mathfrak{P}_1) = f(r_1 + 3, 1, r_1)$, $f(\mathfrak{P}_2) = f(r_2 + 2, 1, r_2)$, and $\mathfrak{P}_1$ is realized on the horizontal axis, $\mathfrak{P}_2$ on the vertical axis. Furthermore, we have $r_1 + r_2 \leqq 4$, and dominant members are not allowed. Therefore, the possible intersection graphs of $\mathfrak{P}_1$ and $\mathfrak{P}_2$ are easily seen to be the graphs depicted in Fig. 2.

Following the procedure described in part (i) of Theorem 2.1, we now obtain precisely seven intersection types of extremal families, representatives of which (using rectangles and straight line segments) are shown in Fig. 3. The label $\mathfrak{P}_{ij}$ indicates that the intersection graph of $\mathfrak{P}_1$ is G_i and that of $\mathfrak{P}_2$ is H_j. The shading

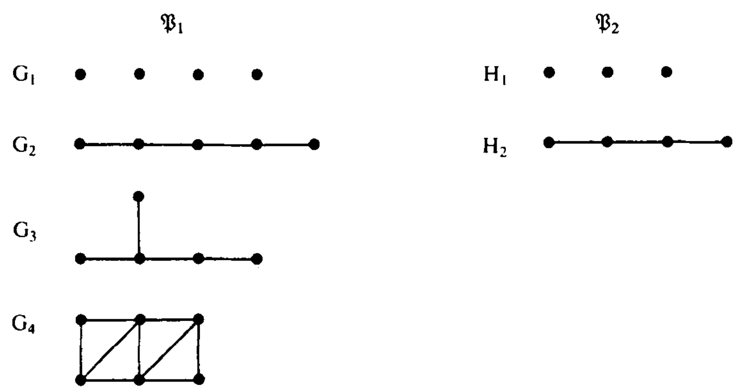

Fig. 2.

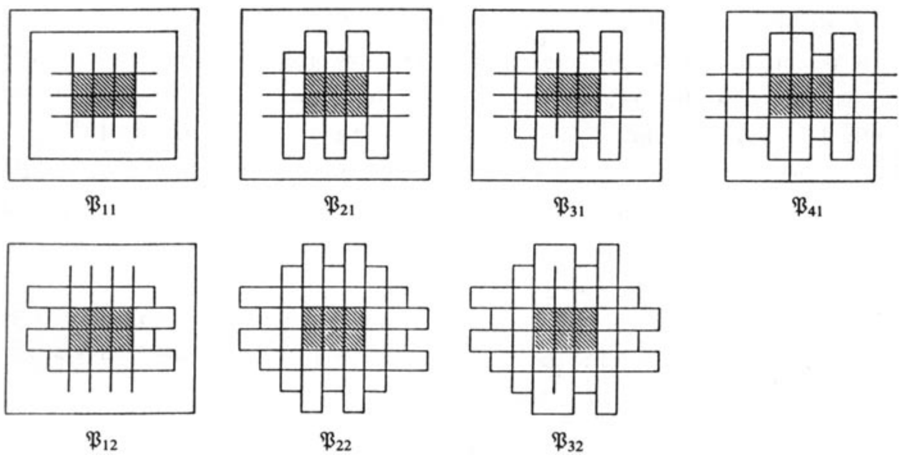

Fig. 3.

will be explained in Section 3. Notice that $\mathfrak{P}_{11}$ has two dominant members, while each of $\mathfrak{P}_{21}$, $\mathfrak{P}_{31}$ and $\mathfrak{P}_{12}$ has one dominant member. The family $\mathfrak{P}_{11}$ represents the type of the Kalai family $\mathfrak{C}(9,2,4)$.

3. Geometric properties of extremal families

Before we can proceed to a detailed description of extremal families, it is necessary to recall from Part I the notion of an exposed member of a family of boxes in R^d. When $d = 1$, the following definition coincides with that given for families of intervals in Section 2.

Let $\mathfrak{P}$ be a family of boxes, $A \in \mathfrak{P}$, and assume first that A has non-empty interior. We say that A is *exposed* in $\mathfrak{P}$ if there exists a supporting hyperplane H_A of A which is parallel to some coordinate hyperplane H_i and such that the closed half-space bounded by H_A and containing A does not contain any member of $\mathfrak{P}$ in its interior. In other words, every $P \in \mathfrak{P}$ either meets H_A or lies in the complementary open half-space bounded by H_A. In the latter case we shall say that P is *separated* from A by H_A. The hyperplane H_A is called an *exposed hyperplane* associated with A. Next assume that A lies in some hyperplane. In this case, A is automatically exposed in $\mathfrak{P}$ and any hyperplane containing A and parallel to some coordinate hyperplane is an exposed hyperplane associated with A. Clearly, $\mathfrak{P}$ has at least one exposed member unless all the sets in $\mathfrak{P}$ are copies of R^d.

We now turn to *extremal* families in R^d. The following assertions reveal the rather stringent geometrical conditions that a family of boxes must satisfy in order to be extremal. Most of these conditions follow easily "by inspection" from the construction described in Theorem 2.1 and the properties of extremal interval graphs listed in (2.1) and (2.2). We restrict our attention to families of the form $\mathfrak{P} = \mathfrak{P}_1 \times \cdots \times \mathfrak{P}_d$ as defined in part (i) of Theorem 2.1. In view of part (ii), this is no loss of generality. The f-vector of $\mathfrak{P}$ is $f(n,d,r)$; the parameters $p, q, r_1, \ldots, r_n$ and the index set J are explained in the preceding section. Here we shall assume that $n \geqq d + r$; otherwise, the intersection type of $\mathfrak{P}$ is uniquely determined and indeed very simple (see (2.5) above). The reader may wish to visualize the arguments to follow by examining the examples of extremal families shown in Fig. 3.

Our first assertion reads as follows.

(3.1) *There exist precisely* $2d$ *exposed members of* $\mathfrak{P}$. *For each* $i = 1, \ldots, d$, *two exposed members (called opposite) are strictly separated by a hyperplane parallel to* H_i.

This follows from (2.1) on noting that the exposed members of $\mathfrak{P}$ are precisely the sets $\tau_i^{-1}(P_i)$, where $i \in \{1, \ldots, d\}$ and P_i is exposed in $\mathfrak{P}_i$. Moreover, $\mathfrak{P}_i$ has at least $r_i + 1$ members; this is a consequence of $n \geqq d + r$, i.e., $p \geqq 2$.

(3.2) *The exposed members of* $\mathfrak{P}$ *completely surround a compact box with interior points* $B_{\mathfrak{P}}$.

By "completely surround" we mean the following. Let U denote the union of the $2d$ exposed members of $\mathfrak{P}$. Then among the connected components of $R^d \setminus U$, exactly one is bounded; $B_{\mathfrak{P}}$ is the closure of that component.

In view of the previous assertion, it is enough to observe that any two exposed members of $\mathfrak{P}$ intersect unless they are opposite to one another. This is clear from the construction of $\mathfrak{P}$. In Fig. 3, the respective rectangles $B_{\mathfrak{P}}$ are shaded.

(3.3) *Each point of* $B_{\mathfrak{P}}$ *lies in at least* $r - d$ *members of* $\mathfrak{P}$, *and each vertex of* $B_{\mathfrak{P}}$ *lies in precisely* r *members.*

Again, this follows from the corresponding one-dimensional result (2.1). The number of sets in $\mathfrak{P}$ containing a given point of $B_{\mathfrak{P}}$ equals at least

$$\sum_{i=1}^{d} (r_i - 1) + r - \sum_{i=1}^{d} r_i = r - d,$$

and this bound is sharp. A similar argument yields the second statement.

For the next assertion we have to assume that $r \geqq d + 1$.

(3.4) *If $r-d-1$ or fewer members of $\mathfrak{P}$ are removed from $\mathfrak{P}$, then the union of the remaining members is still contractible. Also, for $i=0,\dots,r-d-1$, the set of all points contained in at least $i+1$ members of $\mathfrak{P}$ is contractible.*

This is an easy deduction from assertion (3.3). The number $r-d-1$ is the largest for which the stated properties hold. Incidentally, the fact that the set of points covered by at least $i+1$ members of $\mathfrak{P}$ has Euler characteristic 1 is equivalent to

$$\sum_{k\geqq i}(-1)^{k-i}\binom{k}{i}f_k(n,d,r)=1$$

(see [2], Hilfssatz 5). The contractibility of that set is, of course, a much stronger property.

(3.5) *Without altering the intersection graph $G(\mathfrak{P})$, the members of $\mathfrak{P}$ can be modified (if necessary) so that their union becomes a compact box.*

In fact, in view of (3.2) and the very definition of exposed sets, it suffices to replace each member of $\mathfrak{P}$ by its intersection with $B_{\mathfrak{P}}$. Thus the compact box in question can be chosen to be $B_{\mathfrak{P}}$.

The next result will be essential to the inductive proof of Theorem 2.1. Suppose $d>1$, and recall from Part I (or directly from the definition of exposed sets) that if A is exposed in $\mathfrak{P}$, then

$$\mathfrak{P}_A=\{P\cap A\,|\,P\in\mathfrak{P}\backslash A,\ P\cap A\neq\emptyset\}$$

can be be regarded as a family of boxes in R^{d-1}. Simply identify R^{d-1} with the exposed hyperplane H_A associated with A. Any non-empty intersection of members of $\mathfrak{P}$ meeting A must necessarily meet H_A. We have

(3.6) *For each exposed member A of $\mathfrak{P}$, the family $\mathfrak{P}_A$ is extremal in R^{d-1}. If H_A is parallel to H_{i_0}, say, then*

$$f(\mathfrak{P}_A)=\begin{cases}f(n-p-1,d-1,r-1), & \text{if } i_0\in J,\\ f(n-p,d-1,r-1), & \text{if } i_0\notin J.\end{cases}$$

Moreover, if $P\in\mathfrak{P}$ and $P\cap A=\emptyset$, then $P\cap H_A=\emptyset$.

To prove assertion (3.6), let $P\in\mathfrak{P}$ and assume first that $P\cap A=\emptyset$. From the construction of $\mathfrak{P}$ it follows that $\tau_{i_0}(P)\in\mathfrak{P}_{i_0}$, whence $\tau_{i_0}(P)\cap\tau_{i_0}(A)=\emptyset$ and so $P\cap H_A=\emptyset$. On the other hand, if $P\cap A\neq\emptyset$, $P\neq A$, and $\tau_{i_0}(P)\in\mathfrak{P}_{i_0}$, then

$P \cap A$ is dominant in $\mathfrak{P}_A$. By (2.1), the number of sets with the latter property is $r_{i_0} - 1$. Since $\mathfrak{P}_{i_0}$ has $p + r_{i_0}$ or $p + r_{i_0} - 1$ members, according as $i_0 \in J$ or $i_0 \notin J$, the number of sets in $\mathfrak{P}_A$ turns out to be $n - p - 1$ or $n - p$. Again by construction, $\mathfrak{P}_A$ is extremal in H_A and satisfies $f_{r-1}(\mathfrak{P}_A) = 0$. This yields the assertion. Incidentally, it follows that

$$G(\mathfrak{P}_A) \cong G(\mathfrak{P}_1 \times \cdots \times \hat{\mathfrak{P}}_{i_0} \times \cdots \times \mathfrak{P}_d),$$

where the hat indicates that $\mathfrak{P}_{i_0}$ is to be deleted.

Notice that (3.6) greatly extends assertion (I.2.5) where it was assumed that $n - r$ is a multiple of d. In that case, $q = 0$, and each exposed member of $\mathfrak{P}$ of meets exactly

$$\left[\frac{d-1}{d}(n-r)\right] + r - 1 = n - p$$

other members of $\mathfrak{P}$.

We point out that the family $\mathfrak{Q} = \mathfrak{P} \backslash A$ need not be extremal in R^d. One verifies without difficulty that $\mathfrak{Q}$ is extremal if, and only if, either $q = 0$ or else $q > 0$ and $i_0 \in J$.

Our last assertion shows that—loosely speaking—extremal families of boxes are built up in shells. This statement can be made precise, as follows.

(3.7) *If $\mathfrak{P}'$ is obtained from $\mathfrak{P}$ by removing the $2d$ exposed members of $\mathfrak{P}$, then $\mathfrak{P}'$ is extremal and satisfies*

$$f(\mathfrak{P}') = \begin{cases} f(n-2d, d, n-2d), & \text{if } r + d \leqq n < r + 2d, \\ f(n-2d, d, r), & \text{if } n \geqq r + 2d. \end{cases}$$

Once again, this is a consequence of Theorem 2.1 and the corresponding result (2.2) for the line. Observe that $n \geqq r + 2d$ means $p \geqq 3$, whence for each i, the number of sets in $\mathfrak{P}_i$ is at least $r_i + 2$. Therefore, (2.2) can be applied to both exposed members of $\mathfrak{P}_i$. On the other hand, $r + d \leqq n < r + 2d$ means $p = 2$, and by inspecting all the possibilities one finds that $\mathfrak{P}'$ has non-empty intersection.

The process of dismantling $\mathfrak{P}$, that is, stripping $\mathfrak{P}$ of its shell of exposed members, can be repeated as long as the equivalent of the condition $n \geqq r + d$ holds.

4. Proof of the main theorem

We can now prove Theorem 2.1.

As in the preceding sections, we assume that $n \geqq r \geqq d$, and we define p and q by letting $n - r + d = pd + q$, $0 \leqq q < d$. The meaning of $r_1, \ldots, r_d$ and J is explained in Section 2. Recall that τ_i denotes the orthogonal projection which maps R^d onto the x_i axis, for $i = 1, \dots, d$.

1. First we show that the family of boxes $\mathfrak{P}_1 \times \cdots \times \mathfrak{P}_d$ whose construction is described in part (i) of Theorem 2.1 is indeed extremal in R^d. To begin with, the number of sets in $\mathfrak{P}_1 \times \cdots \times \mathfrak{P}_d$ equals

$$\begin{aligned} f_0(\mathfrak{P}) &= \sum_{i=1}^{d} f_0(\mathfrak{P}_i) + r - \sum_{i=1}^{d} r_i \\ &= \sum_{i \in J} (p + r_i) + \sum_{i \notin J} (p + r_i - 1) + r - \sum_{i=1}^{d} r_i \\ &= qp + (d - q)(p - 1) + r \\ &= pd + q + r - d = n, \end{aligned}$$

as required. Since $f_{r_i}(\mathfrak{P}_i) = 0$ for each i, no more than r members of $\mathfrak{P}_1 \times \cdots \times \mathfrak{P}_d$ can have a common point. Hence $f_r(\mathfrak{P}_1 \times \cdots \times \mathfrak{P}_d) = 0$. On the other hand, applying formula (1.1) to the f-vector of $\mathfrak{P}_i$ we find that

$$f_{r_i - 1}(\mathfrak{P}_i) = \begin{cases} p + 1, & \text{if } i \in J, \\ p, & \text{if } i \notin J, \end{cases}$$

and therefore

$$f_{r-1}(\mathfrak{P}_1 \times \cdots \times \mathfrak{P}_d) = (p + 1)^q p^{d-q}.$$

But clearly the same relation holds for the Kalai family $\mathfrak{C}(n, d, r)$. Hence from Theorem 1.1 we deduce that $f(\mathfrak{P}) = f(\mathfrak{C}(n, d, r))$, as claimed.

2. Conversely, let $\mathfrak{P}$ be a family of boxes in R^d whose f-vector equals $f(n, d, r)$. We shall prove by induction on d and n that $\mathfrak{P}$ is of the type described in part (i) of Theorem 2.1 and, moreover, the $\mathfrak{P}_i$ appearing there can be chosen so as to satisfy part (ii). If $d = 1$, there is of course nothing to prove. If $n = r$, then $\mathfrak{P}$ has nonempty intersection; this case is easily dealt with and is left to the reader. So let us assume that $d > 1$ and $n > r$. We have shown in Part I that $\mathfrak{P}$ contains an exposed member B, say, such that the families $\mathfrak{Q} = \mathfrak{P} \backslash B$ and

$$\mathfrak{P}_B = \{P \cap B \mid P \in \mathfrak{Q},\ P \cap B \neq \emptyset\}$$

satisfy

$$f(\mathfrak{Q}) = f(n - 1, d, r) \tag{4.1}$$

and

$$f(\mathfrak{P}_B) = \begin{cases} f(n-p, d-1, r-1), & \text{if } q = 0, \\ f(n-p-1, d-1, r-1), & \text{if } q > 0. \end{cases} \tag{4.2}$$

This follows from the proof of inequalities (I.3.8) and (I.3.9) and the fact that if $\mathfrak{P}$ is extremal in R^d, then equality must hold in these inequalities for all k. (Here we have expressed the right-hand side of (I.3.9) in terms of p and q and have slightly changed the notation.) Thus both $\mathfrak{Q}$ and $\mathfrak{P}_B$ are extremal. Because of assertion (I.2.2), we may also assume that

$$P \in \mathfrak{P},\ P \cap B = \varnothing \Rightarrow P \cap H_B = \varnothing, \tag{4.3}$$

where H_B denotes the exposed hyperplane associated with B. This allows us to regard $\mathfrak{P}_B$ as a family of boxes in R^{d-1}.

Let us first remark upon a particular case, i.e., $n \leqq d + r$. In view of what we have said in Section 2, it suffices to show that the intersection graph of $\mathfrak{P}$ is isomorphic to the graph in (2.5). This in turn follows easily from $f(\mathfrak{P}_B) = f(n-2, d-1, r-1)$, by induction on d. For the remainder of the proof we shall assume, therefore, that $n > d + r$.

Define the index i_0 by requiring that H_{i_0} be parallel to H_B. There are two main cases to consider.

Case 1: $q = 0$

Since $n - 1 - r + d = (p-1)d + d - 1$, the induction hypothesis implies that there exist families of intervals $\mathfrak{Q}_1, \ldots, \mathfrak{Q}_d$ and an index $j_0 \in \{1, \ldots, d\}$ such that

$$G(\mathfrak{Q}) \cong G(\mathfrak{Q}_1 \times \cdots \times \mathfrak{Q}_d) \tag{4.4}$$

and

$$f(\mathfrak{Q}_i) = \begin{cases} f(p + s_i - 1, 1, s_i), & \text{if } i \neq j_0, \\ f(p + s_i - 2, 1, s_i), & \text{if } i = j_0, \end{cases} \tag{4.5}$$

where the s_i are positive integers satisfying

$$s_1 + \cdots + s_d \leqq r. \tag{4.6}$$

Note here that $\{1, \ldots, d\} \backslash \{j_0\}$ plays the role of the index set J in Theorem 2.1. Further, we may assume that the $\mathfrak{Q}_i$ do not contain dominant members and that if $Q_i \in \mathfrak{Q}_i$ and $Q \in \mathfrak{Q}$ correspond to each other under the isomorphism (4.4), then $Q_i = \tau_i(Q)$.

We claim that $j_0 = i_0$. To prove this, choose A to be the exposed member of $\mathfrak{Q}$ whose associated exposed hyperplane H_A is parallel to H_B and such that B lies in the closed half-space bounded by H_A and not containing any member of $\mathfrak{Q}$ in its interior. Since $n - 1 \geqq d + r$ and the induction hypothesis applies to $\mathfrak{Q}$, assertion (3.1) shows that A is indeed uniquely determined. We deduce from (4.5) and (3.6) that

$$(4.7) \qquad f(\mathfrak{Q}_A) = \begin{cases} f(n-p-1, d-1, r-1), & \text{if } i_0 \neq j_0, \\ f(n-p, d-1, r-1), & \text{if } i_0 = j_0, \end{cases}$$

where the definition of $\mathfrak{Q}_A$ is completely analogous to that of $\mathfrak{P}_B$ above. Notice that $\tau_{i_0}(A) \in \mathfrak{Q}_{i_0}$ and, in fact, $\tau_{i_0}(A)$ is exposed in $\mathfrak{Q}_{i_0}$.

The last statement of assertion (3.6), applied to $\mathfrak{Q}$, reads

$$(4.8) \qquad Q \in \mathfrak{Q},\ Q \cap A = \varnothing \Rightarrow Q \cap H_A = \varnothing.$$

This means that of all the members of $\mathfrak{Q}$, only A and the neighbors of A in $\mathfrak{Q}$ can intersect B. Assume, for the moment, that $j_0 \neq i_0$. We know from (4.2) that B meets $n - p$ other members of $\mathfrak{Q}$. In view of (4.7), this implies that B meets A and all the $n - p - 1$ neighbors of A in $\mathfrak{Q}$. Hence $\mathfrak{P}$ and $\mathfrak{Q} \cup \{A \cap B\}$ have the same intersection graph, and $A \cap B$ is dominant in $\mathfrak{Q}$. As a result, $f_k(\mathfrak{P}_B) = f_k(\mathfrak{Q}_A) + f_{k-1}(\mathfrak{Q}_A)$ for all k, with the convention $f_{-1}(\mathfrak{Q}_A) = 1$. Since $f(\mathfrak{Q}_A) = f(n-p-1, d-1, r-1)$, these relations easily imply that $f(\mathfrak{P}_B) = f(n-p, d-1, r)$, contradicting (4.2). So we must have $j_0 = i_0$, as asserted. As a consequence, the number of neighbors of A in $\mathfrak{Q}$ turns out to be $n - p$. We now distinguish three subcases of Case 1. (These subcases are illustrated in Fig. 1.)

(a) *B meets all the neighbors of A in $\mathfrak{Q}$, but not A itself.*

It follows from (4.3) that $\tau_{i_0}(B) \cap \tau_{i_0}(A) = \varnothing$. On the other hand, $\tau_{i_0}(B)$ meets all the neighbors of $\tau_{i_0}(A)$ in $\mathfrak{Q}_{i_0}$.

(b) *B meets A and all the neighbors of A in $\mathfrak{Q}$ except one, C say, which is not dominant in $\mathfrak{Q}$.*

We claim that $\tau_{i_0}(C) \in \mathfrak{Q}_{i_0}$. Assume, to the contrary, that $\tau_i(C) \in \mathfrak{Q}_i$ for some $i \neq i_0$. Let X and Y be the opposite exposed members of $\mathfrak{Q}$ which are strictly separated by a hyperplane parallel to H_i. That X and Y exist follows from assertion (3.1) applied to $\mathfrak{Q}$. Assertion (3.3) shows that both $X \cap A$ and $Y \cap A$ contain points which lie in r different members of $\mathfrak{Q}$. Since C is not dominant in $\mathfrak{Q}$, it cannot intersect both X and Y. Say it misses X. Then B intersects A and all the neighbors of A in $\mathfrak{Q}$ meeting X. Hence B must include points of $X \cap A$ which are contained in r members of $\mathfrak{Q}$. These points would then lie in $r + 1$ members of $\mathfrak{P}$, which is impossible. We conclude that $\tau_{i_0}(B)$ meets $\tau_{i_0}(A)$ and all the neighbors of $\tau_{i_0}(A)$ in $\mathfrak{Q}_{i_0}$ except one, the exception being $\tau_{i_0}(C)$.

(c) *B meets A and all the neighbors of A in $\mathfrak{Q}$ except one, C say, which is dominant in $\mathfrak{Q}$.*

Again by (4.3), we have $\tau_{i_0}(B) \cap \tau_{i_0}(C) = \emptyset$. Because C is dominant in $\mathfrak{Q}$, $\tau_{i_0}(C)$ intersects all members of $\mathfrak{Q}_{i_0}$, whereas $\tau_{i_0}(B)$ meets $\tau_{i_0}(A)$ and all the neighbors of $\tau_{i_0}(A)$ in $\mathfrak{Q}_{i_0}$ (and no other member). Note, however, that $\tau_{i_0}(C) \notin \mathfrak{Q}_{i_0}$.

We now set

$$\mathfrak{P}_{i_0} = \mathfrak{Q}_{i_0} \cup \begin{cases} \{\tau_{i_0}(B), \tau_{i_0}(A)\}, & \text{if case (c) arises,} \\ \{\tau_{i_0}(B)\}, & \text{otherwise.} \end{cases}$$

It follows from assertion (2.3) and the foregoing discussion that $\mathfrak{P}_{i_0}$ is extremal and does not contain a dominant set. Moreover,

$$f(\mathfrak{P}_{i_0}) = f(p + r_{i_0} - 1, 1, r_{i_0}),$$

where

$$r_{i_0} = \begin{cases} s_{i_0} + 1, & \text{if case (c) arises,} \\ s_{i_0}, & \text{otherwise.} \end{cases}$$

Letting $\mathfrak{P}_i = \mathfrak{Q}_i$ and $r_i = s_i$ for $i \neq i_0$ and using (4.4), we conclude that

$$G(\mathfrak{P}) \cong G(\mathfrak{P}_1 \times \cdots \times \mathfrak{P}_d),$$

as required. There remains only to check whether $r_1 + \cdots + r_d \leqq r$ is satisfied. But in view of (4.6), there is nothing to prove unless we are in case (c). In that case, we must have $s_1 + \cdots + s_d < r$ because otherwise $\mathfrak{Q}$ would not contain a dominant member. Thus in spite of $r_{i_0} = s_{i_0} + 1$, $r_1 + \cdots + r_d \leqq r$ is still true. Finally, for $i = 1, \ldots, d$, each set in $\mathfrak{P}_i$ is the image under τ_i of the corresponding set in $\mathfrak{P}$. This completes the proof of Theorem 2.1 in Case 1.

Case 2: $q > 0$

The reasoning in this case is completely similar to that used in Case 1. As we have now $n - 1 - r + d = pd + q - 1$, the induction hypothesis implies that there exist families of intervals $\mathfrak{Q}_1, \ldots, \mathfrak{Q}_d$ which satisfy (4.4) and, unlike Case 1,

$$f(\mathfrak{Q}_i) = \begin{cases} f(p + s_i, 1, s_i), & \text{if } i \in J_0, \\ f(p + s_i - 1, 1, s_i), & \text{if } i \notin J_0. \end{cases} \tag{4.9}$$

Here the s_i are positive integers such that (4.6) holds, and $J_0 \subset \{1,\ldots,d\}$ is a set of $q-1$ indices. The $\mathfrak{Q}_i$ do not have dominant members, and each interval in $\mathfrak{Q}_i$ is the image under τ_i of the corresponding box in $\mathfrak{Q}$. We now choose A to be an exposed member of $\mathfrak{Q}$ and H_A to be the exposed hyperplane associated with A exactly as we did in Case 1. The induction hypothesis applies to $\mathfrak{Q}$ and so assertions (3.6) and (4.9) yield

$$f(\mathfrak{Q}_A) = \begin{cases} f(n-p-2, d-1, r-1), & \text{if } i_0 \in J_0, \\ f(n-p-1, d-1, r-1), & \text{if } i_0 \notin J_0, \end{cases} \tag{4.10}$$

where, as before, i_0 is defined by requiring H_{i_0} to be parallel to H_A. From (4.10) we deduce that $i_0 \notin J_0$ in much the same manner as we proved $i_0 = j_0$ in Case 1. Thus the number of neighbors of A in $\mathfrak{Q}$ is equal to $n-p-1$. Again, we have to consider three subcases (a),(b),(c) of Case 2, depending on whether B and A are disjoint, or B meets A but misses some neighbor C of A in $\mathfrak{Q}$, where C is either non-dominant or dominant in $\mathfrak{Q}$. The argument in each subcase is analogous to that used in the corresponding subcase of Case 1 and will be omitted. This concludes the proof of Theorem 2.1. ■

We wish to point out that each of the subcases considered above can actually occur. This is demonstrated by the examples of extremal families shown in Fig. 3. (Here $p = 3$ and $q = 1$, so we are in Case 2.) In order to satisfy (4.1) and (4.2), we take B to be the left-hand exposed rectangle of $\mathfrak{P}_{ij}$ whose associated exposed support line H_B is vertical. Subcase (a) of Case 2 arises when the family in question is $\mathfrak{P}_{11}$, $\mathfrak{P}_{31}$, $\mathfrak{P}_{12}$, or $\mathfrak{P}_{32}$, and subcase (b) arises when the family is $\mathfrak{P}_{21}$ or $\mathfrak{P}_{22}$. Finally, subcase (c) arises when the family is $\mathfrak{P}_{41}$.

5. The number of extremal families

This final section is concerned with enumerating the intersection types of extremal families of boxes in R^d.

In the special case $d = 1$, the problem of counting extremal families was solved in our recent paper [5] on extremal interval graphs. The number $A_{n,r}$ of such graphs having n vertices and clique number r will be considered a "known" function in what follows.

In view of the remarks made in Section 2, it suffices to enumerate extremal families of the form $\mathfrak{P}_1 \times \cdots \times \mathfrak{P}_d$ as defined in part (i) of Theorem 2.1. We shall first show that essentially different collections of families $\mathfrak{P}_1,\ldots,\mathfrak{P}_d$ give rise to distinct families $\mathfrak{P}_1 \times \cdots \times \mathfrak{P}_d$. More precisely, we shall prove

THEOREM 5.1. *Let $\mathfrak{P}_1, \ldots, \mathfrak{P}_d$ and $\mathfrak{Q}_1, \ldots, \mathfrak{Q}_d$ be two collections of extremal families of intervals which both satisfy the hypotheses of part* (i) *of Theorem* 2.1. *Assume that $G(\mathfrak{P}_1 \times \cdots \times \mathfrak{P}_d) \cong G(\mathfrak{Q}_1 \times \cdots \times \mathfrak{Q}_d)$. Then, after renumbering (if necessary),*

$$G(\mathfrak{P}_i) \cong G(\mathfrak{Q}_i), \qquad i = 1, \ldots, d.$$

PROOF. From the way extremal families are built up, we know that

$$G(\mathfrak{P}_1 \times \cdots \times \mathfrak{P}_d) \cong G(\mathfrak{P}_1) + \cdots + G(\mathfrak{P}_d) + K_v,$$

$$G(\mathfrak{Q}_1 \times \cdots \times \mathfrak{Q}_d) \cong G(\mathfrak{Q}_1) + \cdots + G(\mathfrak{Q}_d) + K_w,$$

where v and w are certain non-negative integers. We claim that $v = w$ and, consequently, $G(\mathfrak{P}_1) + \cdots + G(\mathfrak{P}_d) \cong G(\mathfrak{Q}_1) + \cdots + G(\mathfrak{Q}_d)$. When $n \geqq r + d$, this follows from the fact that the vertices of K_v and K_w are the only dominant vertices of $G(\mathfrak{P}_1 \times \cdots \times \mathfrak{P}_d)$ and $G(\mathfrak{Q}_1 \times \cdots \times \mathfrak{Q}_d)$, respectively. The latter is no longer true when $n < r + d$; however, in this case we have $v = w = r - d$ (see Section 2). Now define a (non-empty) graph G to be *join-irreducible* if one cannot write $G = H + K$ with non-empty graphs H and K. Any non-empty graph is representable as the join of join-irreducible subgraphs. According to Zykov [15] (who uses the term *simple* for join-irreducible), the representation is unique to within a permutation of the subgraphs involved. Indeed, a graph G is join-irreducible if, and only if, its complement $\bar{G}$ is connected. Thus if G had two distinct representations in terms of join-irreducible graphs, then $\bar{G}$ would be the disjoint union of connected graphs in two different ways. This is impossible. So it remains to check whether the intersection graphs $G(\mathfrak{P}_i)$ and $G(\mathfrak{Q}_i)$ above are join-irreducible. But it is well known and easily verified that interval graphs do not contain induced 4-cycles, i.e., 4-cycles without a chord. On the other hand, if the join of two non-empty graphs has no dominant vertex, then it must necessarily contain an induced 4-cycle. This establishes the theorem. ■

Using Theorem 5.1 we can now determine, at least in principle, the numbers of distinct extremal families of boxes having a given f-vector. As already mentioned, these numbers will be expressed in terms of the corresponding numbers $A_{n,r}$ of extremal interval graphs. Unfortunately, the resulting formulas become rapidly cumbersome when the dimension d increases. The reason is that, given n, d and r, all the possible partitions of $r, r-1, \ldots, d$ into d positive integers $r_1, \ldots, r_d$ have to be considered. Therefore, we shall content ourselves with demonstrating the general procedure in the case $d = 2$. Even there, reasonably "nice" expressions cannot be expected.

Let $N(n,d,r)$ denote the number of intersection types of extremal families of boxes in R^d whose f-vector is $f(n,d,r)$. In particular, $N(n,1,r) = A_{n,r}$; these numbers were computed in Section 3 of [5]. If we put

$$B_{n,r} = A_{n,r} - A_{n-1,r-1},$$

with the convention that $A_{n,0} = 0$, then $B_{n,r}$ is the number of extremal interval graphs on n vertices, none of which is dominant, and clique number r.

We can now prove

THEOREM 5.2. *For $n \geqq r \geqq 2$,*

$$N(n,2,r) = \begin{cases} \sum_{i=1}^{[r/2]} \left(B_{t+i,i}(A_{t+r-i,r-i} - A_{t+i,i}) + \binom{B_{t+i,i}+1}{2} \right), & \text{if } n-r=2t, \\ \sum_{i=1}^{r-1} B_{t+i+1,i} A_{t+r-i,r-i}, & \text{if } n-r=2t+1. \end{cases}$$

PROOF. If $n - r = 2t$, then in the notation of Section 1, $p = t + 1$ and $q = 0$. Using Theorem 5.1 and the description of extremal families of the form $\mathfrak{P}_1 \times \mathfrak{P}_2$ in Theorem 2.1, we find that $N(n,2,r)$ equals the number of unordered pairs of extremal interval families $\mathfrak{P}_1, \mathfrak{P}_2$ which do not have dominant members and satisfy

$$f(\mathfrak{P}_i) = f(t + r_i, 1, r_i), \qquad i = 1,2,$$

with $r_1 + r_2 \leqq r$. For a given i with $1 \leqq i \leqq [r/2]$, there are $B_{t+i,i}$ extremal families having f-vector $f(t + i, 1, i)$ and

$$B_{t+i+1,i+1} + \cdots + B_{t+r-i,r-i} = A_{t+r-i,r-i} - A_{t+i,i}$$

such families having f-vector $f(t + j, 1, j)$ for some $j \in \{i + 1, \ldots, r - i\}$. In addition, there are

$$\binom{B_{t+i,i} + 1}{2}$$

unordered pairs of extremal families each having f-vector $f(t + i, 1, i)$. This yields the desired formula in case $n - r$ is even.

Now let $n - r$ be odd, i.e., $n - r = 2t + 1$. Then $p = t + 1$ and $q = 1$, and the reasoning is completely analogous to that used above. For a given i, $1 \leqq i \leqq r - 1$, there are $B_{t+i+1,i}$ extremal interval families without dominant members having f-vector $f(t + i + 1, 1, i)$, and

$$B_{t+1,1} + \cdots + B_{t+r-i,r-i} = A_{t+r-i,r-i}$$

such families having f-vector $f(t+j,1,j)$ for some $j \in \{1,\dots,r-i\}$. This completes the proof of the theorem. ■

The following particular instances of $N(n,2,r)$ are easily verified. Set $k = [n/2]$. Then

$$N(n,2,3) = \begin{cases} B_{k+1,2} + A_{k,2}, & n \text{ even,} \\ A_{k+1,2}, & n \text{ odd,} \end{cases}$$

$$N(n,2,4) = \begin{cases} A_{k+1,3} + \binom{A_{k,2}}{2}, & n \text{ even,} \\ B_{k+2,3} + B_{k+1,3} + A_{k+1,2}A_{k,2}, & n \text{ odd.} \end{cases}$$

Incidentally, it was shown in [5] that

$$A_{k,2} = 2^{k-4} + 2^{[k/2]-2}, \qquad k \geqq 3,$$

$$A_{k,3} = \frac{1}{4}(3^{[k/2]-2} + 1)(3^{[(k+1)/2]-2} + 1), \qquad k \geqq 4.$$

A concise formula for $N(n,2,r)$, with r arbitrary, is highly unlikely. Using the values of $A_{n,r}$ obtained in [5] we have computed $N(n,2,r)$ in the range $2 \leqq r \leqq n \leqq 16$. These numbers are displayed in Table 1. The entries not shown can be

TABLE 1
$N(n,2,r)$

$N(n,2)$	n/r	2	3	4	5	6	7	8
2	2	1						
3	3	1	1					
4	4	1	1	1				
5	5	1	1	1	1			
7	6	1	2	1	1	1		
9	7	1	2	2	1	1	1	
14	8	1	4	3	2	1	1	1
20	9	1	3	7	3	2	1	1
33	10	1	8	7	8	3	2	1
56	11	1	6	23	9	8	3	2
105	12	1	15	25	37	10	8	3
206	13	1	10	79	46	43	10	8
429	14	1	29	70	197	61	44	10
930	15	1	20	265	211	297	65	44
2223	16	1	55	260	1043	386	341	66

found on noting that $N(n,2,r) = N(n-1,2,r-1)$, $r > n/2$, which in turn is a consequence of $B_{n,r} = 0$, $r > n/2$ (see (2.4)). The total number of extremal families of n rectangles in the plane, denoted $N(n,2)$, is also shown.

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